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Orbit Determination for a Lunar Satellite

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Abstract

The orbit determination problem for a lunar satellite is examined from the point of view of geometric determinacy of the data. Various data-type combinations are considered, including radar range, radar range-rate, and optical. Using appropriate coordinate systems, it is shown how the condition equations may be set up and how the number of data points required for solution may be calculated. Finally, it is shown how these considerations are related to the statistical problem of determining the orbit from noisy data.

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Introduction

The problem of determining an orbit for a lunar satellite differs from that for an Earth satellite or a space probe primarily in the geometry involved. Because relative positions of observing stations and satellite are limited, conditions arise in which the orbit may be difficult to establish—difficult in the sense that it is mathematically not well determined. The questions then arise as to how much of the orbit can be determined by the available data, and as to what additional data must be obtained to complete the orbit.

This analysis studies these questions systematically, but without numerical computation.

Formulation of the Problem

As in any other type of orbit determination, the problem for the lunar satellite is solved by an iterative procedure based on differential corrections to an assumed orbit. The orbit is defined by six associated parameters, perhaps the set of osculating elements a, e, i, Ω, ω, χ or perhaps the position and velocity x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ at time $t = 0$. One searches for that orbit which best fits the observations, where "best fit" usually means, in the sense of least squares, that the sum of the squares of the residuals is a minimum.

In general, there will be many more observations than the minimum required for geometrical determination of the six orbital parameters. However, sometimes because of geometric factors, certain of the orbital parameters cannot be found with the available data. This results in the degeneracy of certain matrices used in numerical orbit determination and causes the procedure to break down.

The purpose here is to examine geometric situations which involve this type of degeneracy. To this end it is convenient to evaluate orbit determination from the point of view of the minimum number of observations of a given type or types required to establish an orbit. From the equations for the parameters, it will then be clear as to which are determinate and which are not.

Only the simplest geometric considerations are included. Effects of parallax of stations or of orbit, of non-Keplerian orbit, etc., are neglected. Further, the problem of determining nonorbital parameters is not considered.

Equations

The problem is first set up in full geometric generality, in terms of appropriate coordinate systems. The X, Y, Z coordinate system, shown in Figs. 1 and 2, is nonrotating and has its origin fixed at the center of mass of the Moon. The ξ, η, ζ coordinates are tied to the plane of the satellite orbit, with ξ in the direction of pericenter, η at right angles to ξ in the direction of motion, and ζ perpendicular to ξ, η forming a right-handed system. A third set of coordinates, x, y, z, is tied to the observer-Moon line of sight, so that z is measured along the observer-Moon line, the x, y plane is perpendicular to z, and the origin of coordinates is again at the center of mass of the Moon.

If the probe's coordinates in the ξ, η, ζ system are ξ, η, 0, then the coordinates in the other two systems are given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{B} \begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2}$$

in which $\mathbf{A}$ and $\mathbf{B}$ are the appropriate rotation matrices involving the Euler angles identified in Figs. 1 and 2. Specificially,

$$\mathbf{B} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \tag{3}$$

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$$\mathbf{A} = \begin{pmatrix} \bar{l}_1 & \bar{l}_2 & \bar{l}_3 \\ \bar{m}_1 & \bar{m}_2 & \bar{m}_3 \\ \bar{n}_1 & \bar{n}_2 & \bar{n}_3 \end{pmatrix} \tag{4}$$

with

$$\left.\begin{aligned} l_1 &= \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\ m_1 &= \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i \\ n_1 &= \sin\omega\sin i \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} l_2 &= -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i \\ m_2 &= -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i \\ n_2 &= \cos\omega\sin i \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} l_3 &= \sin\Omega\sin i \\ m_3 &= -\cos\Omega\sin i \\ n_3 &= \cos i \end{aligned}\right\} \tag{7}$$

and corresponding formulas for the components of **A** using barred letters.

Derivatives are obtained by the usual multiplication rule, giving

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{\mathbf{A}}\mathbf{B}\begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix} + \mathbf{A}\dot{\mathbf{B}}\begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix} + \mathbf{AB}\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ 0 \end{pmatrix} \tag{8}$$

Equations (1), (2), and (8) exhibit the observables, $x, y, z, \dot{x}, \dot{y}, \dot{z}$ in terms of the orbit elements. Presumably, if enough values of the observables are obtained (corresponding to observations at different times) then the orbit parameters can be computed.

To complete the representation, the following formulas are also needed:

$$\xi = a(\cos E - e) \qquad \eta = a\sqrt{1 - e^2}\sin E$$

$$\dot{\xi} = -\frac{an\sin E}{1 - e\cos E} \qquad \dot{\eta} = \frac{a\sqrt{1 - e^2}n\cos E}{1 - e\cos E} \tag{9}$$

$$nt + \chi = E - e\sin E \tag{10}$$

in which

E = eccentric anomaly
a = semi-major axis
e = eccentricity
n = mean angular rate
t = time.

The equations above, though appropriate for machine computation, are much too complicated for direct analysis. In the following paragraphs, the equations are simplified, with the hope of extracting some qualitative information. In particular, it is assumed that all observations are made from the same location, that the observer is very far from the orbit, and that the orbital parameters are constant over the period of observation.

In any particular situation, there will be available a number of observations of the satellite, yielding values of observables α_i, β_i, etc., at times t_i. For the present discussion the observables will in general be the coordinates x, y, z and the velocities $\dot{x}, \dot{y}, \dot{z}$ or some subset of these. Each observable gives rise to two equations, the first obtained from Eq. (2) or (8), the second from Eq. (10). It then becomes necessary to determine also the eccentric anomaly, E, at each time point.

Thus, for the orbit determination problem with k values of observables, at τ different times, there will be a system of $k + \tau$ equations to solve. If the number of unknown parameters, p, is greater than $k + \tau$, the system is indeterminate; if p is less than $k + \tau$ it is over-determined and should be solved by statistical procedures; if p is equal to $k + \tau$, it is just determinate (in all cases assuming no degeneracy).

TABLE 1

Minimum Number of Observations for Determining Orbit

Type of Data	Observables	No. of Observations Required
Optical or angle radar	x, y	3
Range radar	z	6 (5)
Doppler radar	$\dot{z}$	6 (5)
Angle radar plus doppler	$x, y, \dot{z}$	2
Range radar plus doppler	$z, \dot{z}$	3

General Comments on Determinacy

How many observations are required to determine an orbit? In the present context, the orbit is presumed to be the osculating orbit, and the orbital elements are presumed to be constant, or at least changing so slowly that for the period of observation, the changes may be neglected. The first step in answering the question, then, is to refer to Eq. (1), (2), and (8).

The term, observation, is taken to mean the set (or subset) of the quantities $x, y, z, \dot{x}, \dot{y}, \dot{z}$ determined at any given instant of time, t. The particular subset associated with an observation depends, of course, on the type of equipment used for the observation. With optical equipment or with angle radar, it may be assumed that the subset consists of x and y. With doppler radar, the subset consists of the single variable, $\dot{z}$, obtained as $\dot{R} - \dot{R}_0$ where R is the range to the observer and R_0 the range from the Moon's center to the observer. Similarly, range radar yields $z = R - R_0$.

To find the number of observations required, it is sufficient, therefore, to note the number of parameters determined by each observation. For optical data, the limitation of two parameters per observation necessitates three observations to establish six orbital elements. For doppler radar data, the requirement would be for six observations to establish all six orbital parameters. However, in some situations only five of the six are determinate. A combination of doppler and angle radar yielding x, y and $\dot{z}$ requires two observations. These requirements, summarized in Table 1, are the minimum for determinacy in each case and imply no redundancy and no degeneracy.

Doppler Data Alone

The minimum of six observations required to determine an orbit using doppler data alone (if the orbit were determinate from such data) must be spaced far enough apart in time to ensure uncorrelated data and far enough apart in position to ensure significant measurements.

When the observations are correlated or when the spatial separation is not adequate, it may take more than six observations to determine a complete orbit, or in some cases only a partial orbit will be determinate. Note, however, that the doppler data curve will immediately yield the period and, hence the semi-major axis of the orbit (see Eqs. (13) and (14)).

Futhermore, it may sometimes be desirable to use the data to determine not only the orbit but also certain quantities that influence the orbit determination. These include, for example, coordinates of the observation stations, instrument biases, gravity field constants, and lunar orbit constants. Of course, for this purpose more data points and better precision are required than for basic orbit determination.

To study the case of doppler data alone, it is convenient to consider in order each of three cases of increasing complexity. In each case it is assumed that there is only one observing station, located far from the observed orbit.

Case 1: Stationary Moon

In this case, there is no loss in generality in choosing the X, Y, Z coordinate axes to be coincident with the x, y, z axes. Thus, $\bar{\imath} = \bar{\omega} = \bar{\Omega} = 0$. Equation (8) then reduces (for the $\dot{z}$-component) to

$$\dot{z} = \frac{an \sin i}{1 - e \cos E} \cdot (-\sin \omega \sin E + \cos \omega \cos E \sqrt{1 - e^2}) \qquad (11)$$

while Eq. (10) remains $nt + \chi = E - e \sin E$.

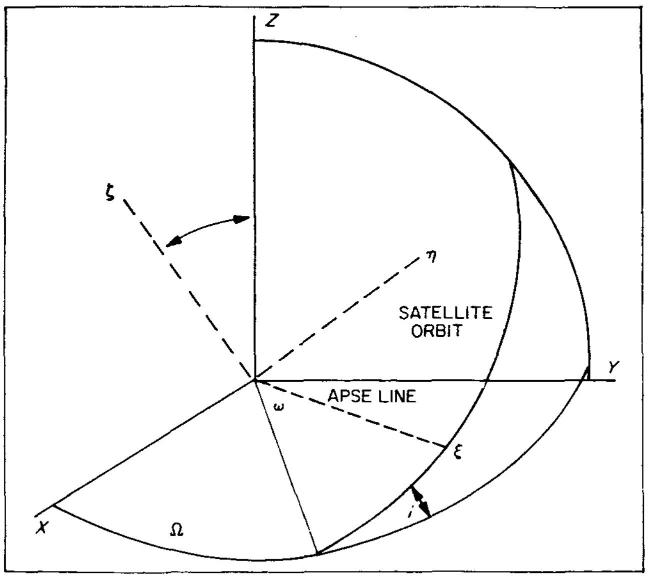

FIG. 1

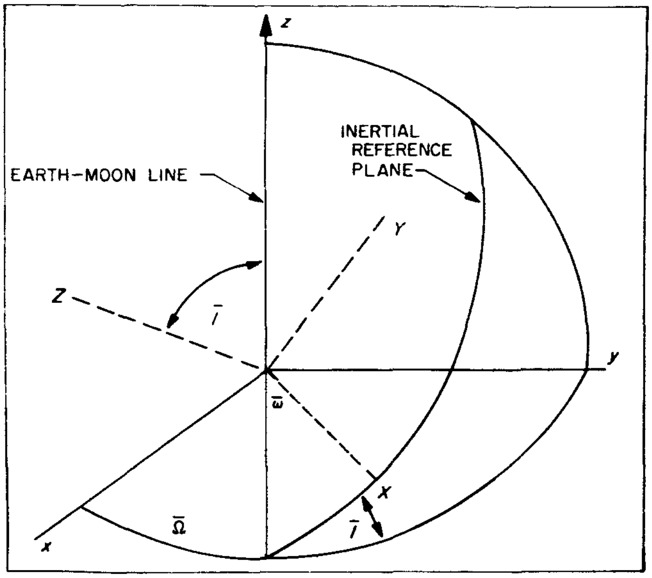

FIG. 2

Here $\dot{z}$ is considered to be the observable (the Moon is so distant that the parallax of the orbit is assumed negligible). The first fact to be noticed in these equations is that the node, Ω, does not appear, and therefore cannot be determined. However, from a superficial look at least, it appears that the remaining five elements a, e, $\sin i$, $\sin \omega$, and χ are uncoupled and can be determined provided only that five uncorrelated measurements at five different times are available. Note that the mean motion, n, is a function of the semi-major axis, a, and that n appears alone in Eq. (10). This fact should decouple the quantities a and $\sin i$.

To summarize, then, it appears that for the stationary Moon, five doppler observations are sufficient to determine an orbit, except for the node, Ω, which is not determinate with any number of doppler observations.

The literature contains several methods of orbit determination using doppler data alone. It may be of interest here to show the relation between the present discussion and one of these methods.

Since doppler data is usually obtained almost continuously, it is of interest to consider the possibility of being able to identify the maxima and minima of $\dot{z}$. To use this type of data, it is convenient to rewrite Eq. (11) in terms of the true anomaly, v, as

$$\dot{z} = \frac{na \sin i}{\sqrt{1 - e^2}} [\cos (v + \omega) + e \cos \omega] \qquad (12)$$

Then, proceeding as in Smart (Ref. 1, p. 359), it is possible to determine e, ω, and $na \sin i$ without resorting to the time equation. If in addition, the gravity constant of the Moon, μ, is known, then by measuring the period, P, and using the relations

$$n = \frac{2\pi}{P} \qquad (13)$$

and

$$a = \mu^{1/3} n^{-2/3} \tag{14}$$

it is possible to evaluate a and $\sin i$ separately. Thus, by using special points on the $\dot{z}$ vs. t curve, it is possible to simplify the computation and obtain some of the orbit elements without working through the time equation and computing eccentric anomaly. On the other hand, not all the information contained in the data is being extracted, for as was first pointed out, all the elements except Ω are determinate, even though the mass of the Moon is unknown.

Case 2: Moon Moving Radially and at Constant Speed with Respect to Observer

Let r be the observer's distance to the satellite, and R his distance to the center of mass of the Moon. Then the analogue to Eq. (11) is

$$\dot{r} = \dot{R} + \frac{na \sin E}{1 - e \cos E} \cdot (-\sin \omega \sin E + \cos \omega \cos E \sqrt{1 - e^2}) \tag{15}$$

in which $\dot{R}$ is a constant. If $\dot{R}$ is assumed known, the problem reduces to Case 1. If $\dot{R}$ is assumed unknown, then one additional observation is required to determine $\dot{R}$.

In contrast to this procedure, the method in Smart (Ref. 1, Fig. 133) uses a graphical computation to find the value of $\dot{R}$.

Case 3: Moon Moving with Angular Velocity $\dot{\theta}$

Choose the x, y, z axes initially coincident with the X, Y, Z axes, and such that the Y-axis points in the direction of motion, the Z-axis in the observer-Moon direction, and the X-axis normal to the plane of motion. Then

$$\dot{\omega} = \dot{\Omega} = 0 \tag{16}$$

and

$$i = \dot{\theta} t \tag{17}$$

The $\dot{z}$ equation from Eq. (8) is

$$\begin{aligned} \dot{z} = \frac{an}{1 - e \cos E} & [-(m_1 \sin \dot{\theta} t + n_1 \cos \theta t) \sin E \\ & + (m_2 \sin \dot{\theta} t + n_2 \cos \dot{\theta} t)\sqrt{1 - e^2} \cos E] \\ & + a\dot{\theta}[(m_1 \cos \dot{\theta} t - n_1 \sin \dot{\theta} t)(\cos E - e) \\ & + (m_2 \cos \dot{\theta} t - n_2 \sin \dot{\theta} t)\sqrt{1 - e^2} \sin E]. \end{aligned} \tag{18}$$

Here m_1 and n_1 are functions of ω, Ω, and i (cf. Eqs. 6 and 7). It is reasonable to expect that all six orbit elements may be determined by the system of Eqs. (18) and (10) provided only that at least six data points are available. However, there remains an uncertainty in the direction of the satellite motion, which is seen as follows:

Assume, for example, that the solution of Eqs. (10) and (18) yields a set of values m_1, m_2, n_1 and n_2. It is easily seen that, though these values determine ω, Ω, and i almost completely, they do allow an ambiguity, which amounts to an uncertainty in the direction of motion. Thus, from Eqs. (6) and (7)

$$\left.\begin{aligned} \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i &= m_1 \\ -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i &= m_2 \\ \sin \omega \sin i &= n_1 \\ \cos \omega \sin i &= n_2 \end{aligned}\right\} \tag{19}$$

can be solved to yield

$$\sin i = \sqrt{n_1^2 + n_2^2} \tag{20}$$

$$\left.\begin{aligned} \sin \omega &= \frac{n_1}{\sqrt{n_1^2 + n_2^2}} \\ \cos \omega &= \frac{n_2}{\sqrt{n_1^2 + n_2^2}} \end{aligned}\right\} \tag{21}$$

$$\begin{aligned} \sin \Omega &= \frac{m_1 n_2 + m_2 n_1}{\sqrt{n_1^2 + n_2^2}} \\ \cos \Omega &= \frac{m_1 n_1 - m_2 n_2}{\sqrt{n_1^2 + n_2^2} \cos i} \end{aligned} \tag{22}$$

in which

$$\begin{aligned} 0 &\leqq i \leqq 180° \\ 0 &\leqq \omega < 360° \\ 0 &\leqq \Omega < 360°. \end{aligned} \tag{23}$$

Equation (20) fails to define the difference between first and second quadrant i. This reflects in the corresponding ambiguity in Ω in Eq. (22).

As a practical matter in orbit determination, the uncertainty in direction should cause no difficulty, since the differential correction procedure presumes the direction of motion known *a priori*.

In summary, then, using six doppler observations on the orbit of a satellite of a moving Moon, it should be possible to completely define the orbit, including all six orbital elements. However, there is an uncertainty in the direction of motion that can be resolved only by *a priori* information, normally available in the differential correction procedure.

Optical Data

Using optical equipment, the primary measurements are angles. By projecting on the x, y plane (Fig. 2) these angles may be interpreted as distances measured in this plane. For the satellite of a distant body, the error introduced by assuming this to be a parallel projection is small, and may be neglected for the present discussion. The difficulty of observing a lunar satellite should, of course, be borne in mind.

Equations (1) and (2) express x and y as functions of the orbital elements of the satellite. Nine equations are obtained from three observations, three each for x, y, and t, where the t equations are of the form of Eq. (10). There are nine unknown quantities in these equations, the six orbital elements a, e, i, ω, Ω, and χ, plus the three values of the eccentric anomaly, E.

Case 1: Stationary Moon

Choose coordinate axes as in Case 1 of the preceding section, so that the equations for x and y from Eq. (2) become

$$\left.\begin{aligned} x &= a[l_1(\cos E - e) + l_2\sqrt{1-e^2}\sin E] \\ y &= a[m_1(\cos E - e) + m_2\sqrt{1-e^2}\sin E] \end{aligned}\right\} \quad (24)$$

and the t equation is

$$nt = -\chi + E - e\sin E \quad (25)$$

In general, Eqs. (24) and (25) can be expected to yield a complete determination of the orbit, except for an ambiguity in the direction of motion. For, assuming l_1, l_2, m_1, and m_2 known, then ω, Ω, and i are to be determined from the system,

$$\left.\begin{aligned} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i &= l_1 \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i &= m_1 \\ -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i &= l_2 \\ -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i &= m_2 \end{aligned}\right\} \quad (26)$$

from which it is possible to solve for $\sin\Omega$ and $\cos\Omega$ to obtain

$$\left.\begin{aligned} \sin\Omega &= \frac{l_2\cos\omega + l_1\sin\omega}{-\cos i} \\ &= -m_2\sin\omega + m_1\cos\omega \\ \cos\Omega &= l_1\cos\omega - l_2\sin\omega \\ &= \frac{m_1\sin\omega + m_2\cos\omega}{+\cos i} \end{aligned}\right\} \quad (27)$$

and from which Ω and ω can be eliminated. Thus, $\cos i$ is found by solving the quadratic equation

$$\begin{aligned} (l_2 m_1 - l_1 m_2)\cos^2 i + (l_2^2 + l_1^2 - m_2^2 - m_1^2)\cos i& \\ - (l_2 m_1 - l_1 m_2) = 0& \end{aligned} \quad (28)$$

Using the value of $\cos i$ so obtained, Eq. (27) then yields values for $\tan\omega$, $\sin\Omega$ and $\cos\omega$. But this leaves the quadrants of Ω and ω uncertain, i.e., the direction of motion is not determined. Thus, for angle data (two angles per observation) with a stationary Moon, the satellite orbit is completely determined by three observations except for the direction of motion. Again, the use of differential corrections yields the additional data necessary to determine the direction of motion.

Case 2: Moon Moving at Constant Angular Velocity $\dot\theta$

Using the coordinates of Case 3 of the preceding section, the equations for the displacements x and y may be written

$$\left.\begin{aligned} x &= X \\ y &= Y\cos\dot\theta t - Z\sin\dot\theta t \end{aligned}\right\} \quad (29)$$

where

$$\left.\begin{aligned} X &= a[l_1(\cos E - e) + l_2\sqrt{1-e^2}\sin E] \\ Y &= a[m_1(\cos E - e) + m_2\sqrt{1-e^2}\sin E] \\ Z &= a[n_1(\cos E - e) + n_2\sqrt{1-e^2}\sin E] \end{aligned}\right\} \quad (30)$$

and

$$nt = -\chi + E - e\sin E \quad (31)$$

The change from Case 1 above is that here the additional quantities n_1 and n_2 are determined. Thus, the exact quadrants of the angles ω and Ω can be found, and the problem becomes completely determinate.

Range Data Plus Range-Rate Data

The appropriate equations in this case are taken from the z component of Eq. (2) and the $\dot z$ component of Eq. (8). For the simplest case, the stationary Moon, the equations reduce to

$$\left.\begin{aligned} z &= a\sin i \\ &\quad\cdot[\sin\omega\cos E + \cos\omega\sin E\sqrt{1-e^2} - e\sin\omega] \\ \dot z &= \frac{an\sin i}{1 - e\cos E} \\ &\quad\cdot[-\sin\omega\sin E + \cos\omega\cos E\sqrt{1-e^2}] \end{aligned}\right\} \quad (32)$$

The significant feature of these z and $\dot z$ equations is that the node angle, Ω, does not appear; thus, the node cannot be determined from range and range-rate data for the case of a stationary Moon.

Since three sets of observations yield nine equations [six from three sets of Eq. (32) and three time equations] for the determination of only eight parameters, it is apparent that the orbit, except for Ω, is sufficiently well determined. Even the quadrant of ω should be obtainable since each of $\sin\omega$ and $\cos\omega$ can be solved for separately.

Both range and range-rate data yield the same type of information as shown in Eq. (32). However, range data has the advantage of yielding, in addition, the absolute distance to the orbit. Thus, the first of Eq. (32) can be written [see Eq. (15)]

$$\begin{aligned} r = R + a\sin i[\sin\omega\cos E& \\ + \cos\omega\sin E\sqrt{1-e^2} - e\sin\omega]& \end{aligned} \quad (33)$$

in which R is the distance to the center of mass of the Moon. Considering R as an additional unknown in the equations, one can then evaluate it as part of the orbit determination procedure.

General Consideration

In the previous sections equations for orbit determination were exhibited in a form meant to show the algebraic or geometric determinacy as simply as possible. No doubt there are many other formulations and other geometric points of view which also reveal the nature of the problem; however, the equations that have been pre-

TABLE 2

Stationary Moon

Type of Data	Observables	No. of Observations	No. of Equations	Variables Determined	Equations	Determinacy
Optical or angle radar	x, y	3	$9 + 1 = 10$	$a, e, \chi, l_1, l_2, m_1, m_2, E_1, E_2, E_3$	$\begin{cases} x = a[l_1 (\cos E - e) + l_2\sqrt{1 - e^2} \sin E] \\ y = a[m_1 (\cos E - e) + m_2\sqrt{1 - e^2} \sin E] \end{cases}$	Yes
Range	z	5	10	$a, e, \chi, n_1, n_2, E_1, E_2, E_3, E_4, E_5$	$z = a[n_1 (\cos E - e) + n_2\sqrt{1 - e^2} \sin E]$	Yes (no Ω)
Doppler	$\dot{z}$	5	10	$a, e, \chi, n_1, n_2, E_1, E_2, E_3, E_4, E_5$	$\dot{z} = \dfrac{an}{1 - e \cos E} (-n_1 \sin E + n_2\sqrt{1 - e^2} \cos E)$	Yes (no Ω)
Angles plus range	x, y, z	2	$8 + 3 = 11$	$a, e, \chi, l_1, l_2, m_1, m_2, n_1, n_2, E_1, E_2$	$\begin{cases} x = a[l_1 (\cos E - e) + l_2\sqrt{1 - e^2} \sin E] \\ y = a[m_1 (\cos E - e) + m_2\sqrt{1 - e^2} \sin E] \\ z = a[n_1 (\cos E - e) + n_2\sqrt{1 - e^2} \sin E] \end{cases}$	No
Angles plus doppler	$x, y, \dot{z}$	2	$8 + 3 = 11$	$a, e, \chi, l_1, l_2, m_1, m_2, n_1, n_2, E_1, E_2$	$\begin{cases} x = a[l_1 (\cos E - e) + l_2\sqrt{1 - e^2} \sin E] \\ y = a[m_1 (\cos E - e) + m_2\sqrt{1 - e^2} \sin E] \\ \dot{z} = an(-n_1 \sin E + n_2\sqrt{1 - e^2} \cos E)/(1 - e \cos E) \end{cases}$	No
Range plus doppler	$z, \dot{z}$	3	9	$a, e, \chi, n_1, n_2, E_1, E_2, E_3$	$\begin{cases} z = a[n_1 (\cos E - e) + n_2\sqrt{1 - e^2} \sin E] \\ \dot{z} = an(-n_1 \sin E + n_2\sqrt{1 - e^2} \cos E)/(1 - e \cos E) \end{cases}$	Yes (no Ω)

Time equation: $nt = -\chi + E - e \sin E$

Auxiliary equations:

$$l_1^2 + m_1^2 + n_1^2 = 1$$
$$l_2^2 + m_2^2 + n_2^2 = 1$$
$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$
$$(l_1 m_2 - l_2 m_1)^2 + 1 = l_1^2 + l_2^2 + m_1^2 + m_2^2$$
$$(m_1 n_1 + m_2 n_2)^2 = (1 - n_1^2 - n_2^2)[n_1^2 + n_2^2 - (m_1 m_2 - m_2 n_1)^2]$$

TABLE 3

Moon in Motion

Type of Data	Observables	No. of Observations	No. of Equations	Variables Determined	Equations	Determinacy
Optical	x, y	3	$9 + 3 = 12$	12 $a, e, \chi, l_1, l_2, m_1, m_2, n_1, n_2, E_1, E_2, E_3$	$\begin{cases} x = a[l_1 (\cos E - e) + l_2\sqrt{1 - e^2} \sin E] \\ y = a[(m_1 \cos \dot{\theta}t - n_1 \sin \dot{\theta}t) (\cos E - e) + (m_2 \cos \dot{\theta}t - n_2 \sin \dot{\theta}t)\sqrt{1 - e^2} \sin E] \end{cases}$	No
Range	z	6	$12 + 1 = 13$	13 $a, e, \chi, m_1, m_2, n_1, n_2, E_1, E_2, E_3, E_4, E_5, E_6$	$z = a[(m_1 \sin \dot{\theta}t + n_1 \cos \dot{\theta}t) (\cos E - e) + (m_2 \sin \dot{\theta}t + n_2 \cos \dot{\theta}t)\sqrt{1 - e^2} \sin E]$	Yes
Range-rate	$\dot{z}$	6	$12 + 1 = 13$	13 $a, e, \chi, m_1, m_2, n_1, n_2, E_1, E_2, E_3, E_4, E_5, E_6$	$\dot{z} = \dfrac{an}{1 - e \cos E} [-(m_1 \sin \dot{\theta}t + n_1 \cos \dot{\theta}t) \sin E + (m_2 \sin \dot{\theta}t + n_2 \cos \dot{\theta}t)\sqrt{1 - e^2} \cos E] + a\dot{\theta}[(m_1 \cos \dot{\theta}t - n_1 \sin \dot{\theta}t) (\cos E - e) + (m_2 \cos \dot{\theta}t - n_2 \sin \dot{\theta}t)\sqrt{1 - e^2} \sin E]$	Yes

sented are direct and do not involve complicated manipulations such as do the methods presented in Smart (Ref. 1).

In conclusion, it is appropriate to indicate how these equations can be implemented to form a tool for estimating quantitatively how well an orbit can be determined. John D. Reichert (Ref. 2) has done just this in a short study in which he computed the correlation matrix and standard deviations over a wide range of conditions for a lunar satellite.

In Reichert's formulation, d_i is considered to be any data type. Then

$$\Delta d_i = \sum \frac{\partial d_i}{\partial \alpha_i} d\alpha_i \qquad (34)$$

where the α_i are the orbital parameters, and the partial derivatives may be obtained from Eqs. (1), (2), (8), and (10). If the matrix of coefficients be denoted by $\mathbf{A}$:

$$\mathbf{A} = \left\| \frac{\partial d_i}{\partial \alpha_i} \right\| \qquad (35)$$

then the character of the matrix $\mathbf{A}$ describes the determinacy of the α_i. In particular, if $\mathbf{A}$ is a square, nonsingular matrix, it will be possible to solve for the $d\alpha_i$. In general, however, there will be many more data points than orbital parameters so that it will be necessary to adopt a statistical approach. In brief, the correlation matrix, $\mathbf{\Lambda}$, for the error estimates in the orbital parameters is given by

$$\mathbf{\Lambda} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{\Lambda}_d\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1} \qquad (36)$$

while the parameters themselves are given by

$$\overrightarrow{\Delta\alpha} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\,\overrightarrow{\Delta d} \qquad (37)$$

in which $\mathbf{A}^T$ means $\mathbf{A}$ transpose, $\mathbf{A}^{-1}$ means $\mathbf{A}$ inverse, $\mathbf{\Lambda}_d$ is the covariance matrix of the observations, $\overrightarrow{\Delta d}$ is the set of observations and $\overrightarrow{\Delta\alpha}$ the set of orbital parameters.

Reichert's study covers the case of a lunar satellite tracked by doppler radar over a time interval ranging from one orbit to one month. This corresponds to the case of Moon in motion in the present discussion.

The results of Reichert's analysis are exhibited in terms of numerical values of elements of the $\mathbf{\Lambda}$ matrix, or equivalently in terms of standard deviation and correlation coefficients for the orbital elements. According to these results it appears that there will be no problem in determining a lunar orbit from doppler data alone, and with reasonable accuracy provided, the doppler data itself is as good as it appears.

Summary

The results of this study are summarized in Table 2 for the stationary Moon and in Table 3 for the Moon in motion. Data combinations not listed may be easily analyzed by referring to their associated equations as listed in the table.

In the first column are listed the data types, with the corresponding observables listed in the second column. The third column shows the minimum number of observations required for an orbit determination, assuming optimum conditions. In the fourth column, the first integer identifies the number of equations arising directly from the observations, including the time equations. The second digit specifies the number of auxiliary equations associated with the relations among the direction cosines l_1, l_2, m_1, m_2, and n_1, n_2.

The Variables-determined column identifies the unknown variables appearing explicitly in the equations. When the direction cosines are determined, the values of ω, Ω, and i may not be determinate. In a physical sense, this usually means that the direction of motion of the satellite is not determinate. The last column indicates determinacy.

References

1. Smart, W. M., *Textbook on Spherical Astronomy*, Cambridge, 4th ed., 1956.
2. Reichert, John D., "Orbit Determination," in *The Behavior of Lunar Satellites and the Determination of Their Orbits*, Ed. by C. R. Gates, JPL, Pasadena, California (TM No. 33-107), September 18, 1962.